

Gauge covariant formulation of Wigner representation through deformational quantization –Application to Keldysh formalism with electromagnetic field–

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We developed a gauge-covariant formulation of the non-equilibrium Green function method for the dynamical and/or non-uniform electromagnetic field by means of the deformational quantization method. Such a formulation is realized by replacing the Moyal product in the so-called Wigner space by the star product, and facilitates the order-by-order calculation of a gauge-invariant observable in terms of the electromagnetic field. An application of this formalism to the linear response theory is discussed.

§1. introduction

The interaction between matter and electromagnetic fields constitutes the most important ingredients of condensed matter physics. Especially, since the high intensity fields become available, the non-linear and non-equilibrium responses are of great current interests. Therefore a microscopic quantum theory for these processes is called for.

There already exist established theories for the response to external perturbations. The Kubo formalism¹⁾ combined with the Matsubara Green function²⁾ is the most standard and successful linear response theory. However, its extension to higher-order responses to the non-equilibrium field is not straightforward. Another method depends on a semiclassical description such as the Boltzmann transport equation.^{3),4)} Although appealing to one's intuition, it sometimes leads to different results from what is obtained from the rigorous Kubo formalism. In contrast to these methods, the Keldysh formalism^{5),6)} yields a powerful theoretical framework for directly handling the non-equilibrium Green function in the presence of electromagnetic field. Kadanoff and Baym introduced the Wigner distribution function (WDF): $f(X, p)$, where $X^\mu = (T, \mathbf{X})$ and $p^\mu = (\omega, \mathbf{p})$ are center-of-mass coordinates and relative energy-momenta, respectively, and derived the equation of motion for the WDF. The WDF is an appropriate function to describe the interacting, many-particle system, and the equation for its time-evolution is called the Quantum Boltzmann Equation (QBE). However, they did not include the electromagnetic field in this formula. Altshuler,⁷⁾ Langreth,⁸⁾ Rammer,⁹⁾ and Mahan^{10),11)} included the

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electromagnetic field in such formula. However, it is not so straightforward to apply these formulas to the non-linear responses.

Recently, we developed a generic and systematic theoretical framework for the linear and non-linear responses under the external constant electromagnetic field.¹³⁾ In that work, we have employed the Wigner representations composed of the set of the center-of-mass time-space coordinates X and gauge covariant momentum $\pi = p - qA(X)$, where p is the momentum, q is the electric charge and $A(X)$ is the electromagnetic potential. Then, we have accomplished the gauge-covariant formulation of the non-equilibrium Green function in the (X, π) space under the constant electromagnetic field, by replacing a conventional product with the Moyal product.¹²⁾ This framework gives a geometric view analogous to the general relativity: the external electromagnetic field is incorporated in the geometry of the Wigner space. Corresponding to the commutation relationship $[\hat{x}^\mu, \hat{\pi}_\nu] = i\hbar\delta^\mu_\nu$ between operators, the Wigner space has the noncommutative geometry between time-space coordinate and energy-momentum. Furthermore, in the presence of the electromagnetic field, the non-zero commutator $[\hat{\pi}_\mu, \hat{\pi}_\nu] = i\hbar q F_{\mu\nu}$ leads also to the noncommutativity within the energy-momentum space. It is highly desirable to generalize this geometrical formulation to the dynamical/non-uniform electromagnetic field. In this paper, we will show that this generalization is achieved by using a deformational quantization method.¹⁴⁾ In the linear order in the electromagnetic field, this formalism completely agrees with the Kubo formalism.¹⁾

This paper is organized as follows. In §2, we begin with a brief review of the Keldysh formalism and the gauge-covariant Dyson equations with external constant electromagnetic fields, which has been developed in the previous paper.¹³⁾ For self-containedness, a brief introduction to the star product and deformational quantization¹⁴⁾ is given in §3. §4 constitutes the main body of the present paper, where we present the gauge-covariant formalism taking into account the generic electromagnetic field using the deformational quantization method. In §5, an application to the linear response theory is given.

§2. Keldysh formalism

In this section, we briefly review the Wigner space of (X, π) and the Moyal product in the case of the constant electromagnetic field, which have been developed in the previous paper.¹³⁾

First, we introduce a matrix of Green functions $\underline{\hat{G}}$ in the Keldysh space,

$$\underline{\hat{G}} = \begin{pmatrix} \hat{G}^R & 2\hat{G}^< \\ 0 & \hat{G}^A \end{pmatrix}, \quad (2.1)$$

where the superscripts R , A , and $<$ denote the retarded, the advanced and the lesser Green functions, respectively. They are defined as

$$\hat{G}^R(x_1, x_2)_{\alpha_1, \alpha_2} := -i\theta(t_1 - t_2)\langle[\hat{\psi}_{\alpha_1}(x_1), \hat{\psi}_{\alpha_2}^\dagger(x_2)]_\mp\rangle, \quad (2.2)$$

$$\hat{G}^A(x_1, x_2)_{\alpha_1, \alpha_2} := i\theta(t_1 - t_2)\langle[\hat{\psi}_{\alpha_1}(x_1), \hat{\psi}_{\alpha_2}^\dagger(x_2)]_\mp\rangle, \quad (2.3)$$

$$\hat{G}^<(x_1, x_2)_{\alpha_1, \alpha_2} := \mp i\langle\hat{\psi}_{\alpha_2}^\dagger(x_2)\hat{\psi}_{\alpha_1}(x_1)\rangle, \quad (2.4)$$

where $\hat{\psi}$ is a Bose (upper sign) or a Fermi (lower sign) field, $\hat{\psi}^\dagger$ is a Hermitian conjugate of $\hat{\psi}$, and $x_1 = (t_1, \mathbf{x}_1)$, α_1 represents an internal degree of freedom. Hereafter, we consider electron systems. The matrix $\hat{\underline{G}}$ of the Green functions satisfies the Dyson equations:

$$\left((\hat{\underline{G}}^{(0)-1} - \hat{\underline{\Sigma}}) * \hat{\underline{G}} \right)_{\alpha_1, \alpha_2} (x_1, x_2) = \delta_{\alpha_1, \alpha_2} \delta(x_1 - x_2), \quad (2.5)$$

$$\left(\hat{\underline{G}} * (\hat{\underline{G}}^{(0)-1} - \hat{\underline{\Sigma}}) \right)_{\alpha_1, \alpha_2} (x_1, x_2) = \delta_{\alpha_1, \alpha_2} \delta(x_1 - x_2), \quad (2.6)$$

where $\hat{\underline{G}}^{(0)}$ is the unperturbed Green function for free electrons (note that the *super-script* $^{(0)}$ means the “unperturbed” in this paper), $\hat{\underline{\Sigma}}$ is the matrix of the self-energies which arises from electron-electron and/or electron-phonon interactions and potential scattering. $(f * g)(x_1, x_2) := \int dx_3 f(x_1, x_3) g(x_3, x_2)$ is the convolution integral.

Next, we introduce the Wigner representation: the center-of-mass coordinates $X^\mu = (T, \mathbf{X}) := (\frac{1}{2}(t_1 + t_2), \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2))$, and the energy-momenta $p^\mu = (\omega, \mathbf{p}) := (\omega_1 - \omega_2, \mathbf{p}_1 - \mathbf{p}_2)$ for relative coordinates.^{9), 10)} Then, the Dyson equations (2.5) and (2.6) are rewritten as

$$\left((\hat{\underline{G}}^{(0)-1} - \hat{\underline{\Sigma}}) \star_{\hbar} \hat{\underline{G}} \right) (X, p) = 1, \quad (2.7)$$

$$\left(\hat{\underline{G}} \star_{\hbar} (\hat{\underline{G}}^{(0)-1} - \hat{\underline{\Sigma}}) \right) (X, p) = 1, \quad (2.8)$$

where the symbol “ \star_{\hbar} ” represents the Moyal product defined by

$$\begin{aligned} (\hat{f} \star_{\hbar} \hat{g})_{\alpha_1, \alpha_2} (X, p) &= \sum_{\alpha_3} \hat{f}_{\alpha_1, \alpha_3} (X, p) e^{\frac{i\hbar}{2} (\overleftarrow{\partial}_{X^\mu} \overrightarrow{\partial}_{p_\mu} - \overleftarrow{\partial}_{p_\mu} \overrightarrow{\partial}_{X^\mu})} \hat{g}_{\alpha_3, \alpha_2} (X, p) \\ &\equiv \sum_{\alpha_3} \hat{f}_{\alpha_1, \alpha_3} (X, p) \star_{\hbar} \hat{g}_{\alpha_3, \alpha_2} (X, p), \end{aligned} \quad (2.9)$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ represent the derivatives which operate only to the left-hand side and the right-hand side, respectively. The average value of the physical observable which is represented by a one-particle operator can be calculated by a one-particle Green function. For example, the particle density $\rho(X)$ and the charge current $J_k(X)$ are written by $G^<(x)$ as

$$\rho(X) = \frac{\hbar}{i} \int \frac{d^{d+1}p}{(2\pi\hbar)^{d+1}} \text{tr} \left[\hat{G}^<(p, X) \right], \quad (2.10)$$

$$J_k(X) = q \frac{\hbar}{i} \int \frac{d^{d+1}p}{(2\pi\hbar)^{d+1}} \text{tr} \left[\left(\nabla_{p_k} \hat{H}(\mathbf{p} - q\mathbf{A}(X)) \right) \hat{G}^<(p, X) \right]. \quad (2.11)$$

Under the uniform and static electromagnetic field $F_{\mu\nu} = \partial_{X^\mu} A_\nu - \partial_{X^\nu} A_\mu = \text{const.}$, we have recently derived the gauge-covariant form of the Dyson equation

$$(\hat{\underline{G}}_0^{(0)-1}(X, \pi) - \hat{\underline{\Sigma}}(X, \pi)) \exp\left(\frac{i\hbar}{2} (\overleftarrow{\partial}_{X^\mu} \overrightarrow{\partial}_{\pi_\mu} - \overleftarrow{\partial}_{\pi_\mu} \overrightarrow{\partial}_{X^\mu} + qF^{\mu\nu} \overleftarrow{\partial}_{\pi^\mu} \overrightarrow{\partial}_{\pi^\nu})\right) \hat{\underline{G}}(X, \pi) = 1, \quad (2.12)$$

$$\hat{G}(X, \pi) \exp\left(\frac{i\hbar}{2}(\overleftarrow{\partial}_{X^\mu} \overrightarrow{\partial}_{\pi_\mu} - \overleftarrow{\partial}_{\pi_\mu} \overrightarrow{\partial}_{X^\mu} + qF^{\mu\nu} \overleftarrow{\partial}_{\pi_\mu} \overrightarrow{\partial}_{\pi_\nu})\right)(\hat{G}_0^{(0)-1}(X, \pi) - \hat{\Sigma}(X, \pi)) = 1, \quad (2.13)$$

where we have used the transformation of variables:^{8),13)}

$$p_\mu - qA_\mu(X) \mapsto \pi_\mu, \quad (2.14)$$

$$f(X, p) \mapsto f(X, \pi). \quad (2.15)$$

Here π_μ is the mechanical momentum in the Wigner representation with the electric charge q of the particle, and the subscript 0 represents the zeroth-order terms with respect to F . All products appearing in the absence of the electromagnetic field is replaced by the Moyal product.¹²⁾ The operator $\exp(\frac{i\hbar}{2}(\overleftarrow{\partial}_{X^\mu} \overrightarrow{\partial}_{\pi_\mu} - \overleftarrow{\partial}_{\pi_\mu} \overrightarrow{\partial}_{X^\mu} + qF^{\mu\nu} \overleftarrow{\partial}_{\pi_\mu} \overrightarrow{\partial}_{\pi_\nu}))$ has been obtained from $\exp(\frac{i\hbar}{2}(\overleftarrow{\partial}_{X^\mu} \overrightarrow{\partial}_{p_\mu} - \overleftarrow{\partial}_{p^\mu} \overrightarrow{\partial}_{X^\mu}))$ by changing the coordinates from p^μ to π^μ and using fact that $F^{\mu\nu} = \text{const.}$ However, with dynamical/non-uniform electromagnetic field, derivative ∂_{X^μ} generates $\partial_{X^\kappa} F^{\mu\nu} (\neq 0)$ which complicates the expression of the product in the (X, π) space. Below we will develop a rigorous treatment of these dynamical/non-uniform cases, which enables for example the description of non-linear responses to even oscillating electromagnetic fields.

§3. Star product and Deformational quantization

To achieve generalization to the case in general electromagnetic fields, we will use the deformational quantization method.¹⁴⁾ It provides the prescription to construct the star product from the usual product $f \cdot g$ and the Poisson bracket $\{f, g\} := \sum_{ij} \alpha^{ij} (\partial_{x^i} f) (\partial_{x^j} g)$. Here, $\{f, g\}$ is assumed to satisfy Jacobi's identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ and the matrix α^{ij} is called the Poisson structure, where f, g and h are smooth functions on a finite-dimensional manifold M . Jacobi's identity can be rewritten by the condition for the Poisson structure:

$$\sum_{l=1}^d \left(\alpha^{il} \partial_{x^l} \alpha^{jk} + \alpha^{jl} \partial_{x^l} \alpha^{ki} + \alpha^{kl} \partial_{x^l} \alpha^{ij} \right) = 0, \quad \text{for } \forall i, j, k = 1 \sim d, \quad (3.1)$$

where we have used $\alpha^{ij} = -\alpha^{ji}$, and d is the dimension of the manifold. The star product is written as

$$f \star g := \sum_{n=0}^{\infty} C_n(f, g) (i\hbar/2)^n, \quad (3.2)$$

where $C_0(f, g) := f \cdot g$ and $C_1(f, g) := \{f, g\}$. Then $C_n(f, g)$ with $n \geq 2$ is defined so as to satisfy the associativity property:

$$f \star (g \star h) = (f \star g) \star h. \quad (3.3)$$

For our purpose, the manifold M is the Wigner space. Kontsevich¹⁴⁾ found an ingenious method to construct the star product, which we introduce here.

Kontsevich's star product

Kontsevich invented the diagram technique to calculate the n -th order term in $i\hbar/2$ for the expansion of $f \star g$:

$$(f \star g)(x) = f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i\hbar}{2} \right)^n \sum_{\Gamma \in G_n} w_{\Gamma} B_{\Gamma, \alpha}(f, g), \quad (3.4)$$

where Γ , $B_{\Gamma, \alpha}(f, g)$ and w_{Γ} are defined as follows:¹⁴⁾

Definition. 1

G_n is a set of the graphs Γ which have $n+2$ vertices and $2n$ edges. Vertices are labeled by symbols "1", \dots , " n ", " L ", " R ". Edges are labeled by symbols (k, v) , where $k = 1, \dots, n$, $v = 1, \dots, n, L, R$ and $k \neq v$. (k, v) means the edge which starts at " k " and ends at " v ". There are two edges starting from each vertex with $k = 1, \dots, n$. L and R are the exception, i.e., they act only as the end points of the edges. Hereafter, V_{Γ} and E_{Γ} represent the set of the vertices and the edges, respectively.

Definition. 2

$B_{\Gamma, \alpha}(f, g)$ is the operator defined as:

$$B_{\Gamma, \alpha}(f, g) := \sum_{I: E_{\Gamma} \rightarrow \{i_1, \dots, i_{2n}\}} \left[\prod_{k=1}^n \left(\prod_{e \in E_{\Gamma}, e=(k, *)} \partial_{I(e)} \right) \alpha^{I((k, v_k^1), (k, v_k^2))} \right] \times \\ \left[\left(\prod_{e \in E_{\Gamma}, e=(*, L)} \partial_{I(e)} \right) f \right] \times \left[\left(\prod_{e \in E_{\Gamma}, e=(*, R)} \partial_{I(e)} \right) g \right], \quad (3.5)$$

where, I is a map from the list of edges $((k, v_k^{1,2}), k = 1, 2, \dots, n)$ to integer numbers $\{i_1, i_2, \dots, i_{2n}\}$. Here, $1 \leq i_n \leq d$ and d is a dimension of the manifold M . $B_{\Gamma, \alpha}(f, g)$ corresponds to the graph Γ in the following way. The vertices "1", \dots , " n ", correspond to the Poisson structure α^{ij} . R and L correspond to the functions f and g , respectively. The edge $e = (k, v)$ represents the derivative acting on the vertex v . Thus the graph Γ corresponds to the bidifferential operator $B_{\Gamma, \alpha}(f, g)$ by the above rules. The simplest diagram for $n = 1$ is shown in Fig. 1 (a), which corresponds to the Poisson bracket $\{f, g\} = \sum_{i_1, i_2} \alpha^{i_1 i_2} (\partial_{x^{i_1}} f)(\partial_{x^{i_2}} g)$. The higher order terms are the

generalizations of this Poisson bracket satisfying Eq. (3.3). Fig. 1 (b) shown a graph Γ with $n = 2$ with the list of edges

$$((1, L), (1, R), (2, R), (2, 3)). \quad (3.6)$$

The operator $B_{\Gamma, \alpha}$ corresponding to this graph is

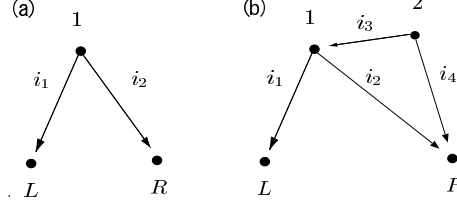


Fig. 1. (a): The graph $\Gamma \in G_1$ corresponding to Poisson bracket. (b): A graph $\Gamma \in G_2$ correspond to the list of edges: $((1, L), (1, R), (2, R), (2, 1)) \mapsto \{i_1, i_2, i_3, i_4\}$.

$$(f, g) \mapsto \sum_{i_1, \dots, i_4} (\partial_{x^{i_3}} \alpha^{i_1 i_2}) \alpha^{i_3 i_4} (\partial_{x^{i_1}} f) (\partial_{x^{i_2}} \partial_{x^{i_4}} g). \quad (3.7)$$

Definition. 3

We put the vertices in the upper-half complex plane $H_+ := \{z \in \mathbf{C} | \text{Im}(z) > 0\}$. R and L are put at 0 and 1, respectively. We associate a weight w_Γ with each graph $\Gamma \in G_n$ as

$$w_\Gamma := \frac{1}{n!(2\pi)^{2n}} \int_{\mathcal{H}_n} \bigwedge_{k=1}^n \left(d\phi_{(k, v_k^1)}^h \wedge d\phi_{(k, v_k^2)}^h \right), \quad (3.8)$$

where ϕ is defined by

$$\phi_{(k, v)}^h := \frac{1}{2i} \text{Log} \left(\frac{(q-p)(\bar{q}-p)}{(q-\bar{p})(\bar{q}-\bar{p})} \right). \quad (3.9)$$

p and q are the coordinates of the vertices “ k ” and “ v ”, respectively. \bar{p} is complex conjugate of p . \mathcal{H}_n represents the space of configurations of n numbered pair-wise distinct points on H_+ :

$$\mathcal{H}_n := \{(p_1, \dots, p_n) | p_k \in H_+, p_k \neq p_l \text{ for } k \neq l\}. \quad (3.10)$$

Here we assume that H_+ has the Poincare metric: $ds^2 = (d(\text{Re}(p))^2 + d(\text{Im}(p))^2) / (\text{Im}(p))^2$, $p \in H_+$. From the metric, we can calculate a geodesic. Note that, $\phi^h(p, q)$ is the angle with the geodesics which is defined by (p, q) and (∞, p) . Namely, $\phi^h(p, q) = \angle pq\infty$. For example, w_Γ corresponding to the Fig. 1 (a) is

$$w_1 = \frac{2}{1!(2\pi)^2} \int_{\mathcal{H}_1} d\frac{1}{2i} \text{Log} \left(\frac{p^2}{\bar{p}^2} \right) \wedge d\frac{1}{2i} \text{Log} \left(\frac{(1-p)^2}{(1-\bar{p})^2} \right) = 1, \quad (3.11)$$

where we have included the factor “2” arisen from the interchange between two edges in Γ . w_Γ corresponding to the Fig. 1 (b) is

$$\begin{aligned} w_{\Gamma(b)} &= \frac{1}{2!(2\pi)^4} \int_{\mathcal{H}_2} d\frac{1}{i} \text{Log} \left(\frac{p_1}{\bar{p}_1} \right) \wedge d\frac{1}{i} \text{Log} \left(\frac{1-p_1}{1-\bar{p}_1} \right) \wedge d\frac{1}{i} \text{Log} \left(\frac{p_2}{\bar{p}_2} \right) \wedge d\frac{1}{2i} \text{Log} \left(\frac{(p_1-p_2)(\bar{p}_1-p_2)}{(p_1-\bar{p}_2)(\bar{p}_1-\bar{p}_2)} \right) \\ &= \frac{1}{2!(2\pi)^4} \int_{\mathcal{H}_2} d\frac{1}{i} \text{Log} \left(\frac{p_1}{\bar{p}_1} \right) \wedge d\frac{1}{i} \text{Log} \left(\frac{1-p_1}{1-\bar{p}_1} \right) \end{aligned}$$

$$\begin{aligned}
 & \wedge d(2 \arg(p_2)) \wedge d|p_2| \frac{\partial}{\partial |p_2|} \frac{1}{2i} \text{Log} \left(\frac{(p_1 - p_2)(\bar{p}_1 - p_2)}{(p_1 - \bar{p}_2)(\bar{p}_1 - \bar{p}_2)} \right) \\
 &= \frac{1}{2!(2\pi)^4} \int_{\mathcal{H}_2} d\frac{1}{i} \text{Log} \left(\frac{p_1}{\bar{p}_1} \right) \wedge d\frac{1}{i} \text{Log} \left(\frac{1 - p_1}{1 - \bar{p}_1} \right) \wedge d\frac{1}{i} \text{Log} \left(\frac{p_2}{\bar{p}_2} \right) \wedge d\frac{1}{i} \text{Log} \left(\frac{1 - p_2}{1 - \bar{p}_2} \right) \\
 &= \frac{w_1^2}{2!} \\
 &= \frac{1}{2},
 \end{aligned} \tag{3.12}$$

where p_1 and p_2 are the coordinates of vertices “1” and “2”, respectively. Here, we have used the following facts:

$$\begin{aligned}
 \int_0^\infty d|p_2| \partial_{|p_2|} \text{Log} \left(\frac{(p_1 - p_2)(\bar{p}_1 - p_2)}{(\bar{p}_1 - p_2)(\bar{p}_1 - \bar{p}_2)} \right) &= \lim_{\Lambda \rightarrow \infty} \text{Log} \left(\frac{(p_1 - \Lambda e^{i \arg(p_2)})(\bar{p}_1 - \Lambda e^{i \arg(p_2)})}{(p_1 - \Lambda e^{-i \arg(p_2)})(\bar{p}_1 - \Lambda e^{-i \arg(p_2)})} \right) \\
 &= \lim_{\Lambda \rightarrow \infty} \text{Log} \left(\frac{(1 - \Lambda e^{i \arg(p_2)})(1 - \Lambda e^{i \arg(p_2)})}{(1 - \Lambda e^{-i \arg(p_2)})(1 - \Lambda e^{-i \arg(p_2)})} \right), \\
 \int_{|p_1| \gtrsim \Lambda} d\text{Log} \left(\frac{p_1}{\bar{p}_1} \right) \wedge d\text{Log} \left(\frac{1 - p_1}{1 - \bar{p}_1} \right) &\xrightarrow{\Lambda \rightarrow \infty} \int_{|p_1| \gtrsim \Lambda} d\text{Log} \left(\frac{p_1}{\bar{p}_1} \right) \wedge d\text{Log} \left(\frac{p_1}{\bar{p}_1} \right) = 0.
 \end{aligned} \tag{3.13}$$

Generally speaking, the integrals are entangled for $n \geq 3$ graphs, and the weight of these are not so easy to evaluate as Eq. (3.12).

§4. Gauge-covariant formulation with electromagnetic fields

In this section, we derive the gauge-covariant Dyson equation. Since we can use the deformational quantization method, we only need C_0 and C_1 in (3.2). We should calculate the zeroth-order and the first-order terms which are given by expanding the Dyson equation (2.6) with respect to \hbar and by the variable transformation $p_\mu - qA_\mu(X) \mapsto \pi_\mu$. The derivation is organized by three steps as shown in Fig 2.

4.1. Transformation of variables and Poisson structure

Step. 1: We derive the zeroth-order and the first-order terms of the Dyson equations with respect to \hbar by the gradient expansion. Expanding the left hand side (LHS) of Eq. (2.5) with respect to \hbar , we obtain the first two terms as

$$((\hat{\underline{G}}_0^{(0)-1}(x) - \hat{\underline{\Sigma}}(x)) \cdot \hat{\underline{G}}(x)), \tag{4.1}$$

$$\frac{i\hbar}{2} \left\{ (\hat{\underline{G}}_0^{(0)-1}(x) - \hat{\underline{\Sigma}}(x)), \hat{\underline{G}}(x) \right\}, \tag{4.2}$$

where $x := (X, p)$ and the dimension of the Wigner space is $4 + 4$. $\{f, g\} := \sum_{i,j} \alpha_0^{ij} (\partial_{x^i} f)(\partial_{x^j} g)$ is the Poisson bracket with the Poisson structure α_0^{ij} ($i, j = 1 \sim 4 + 4$) given by:

$$\alpha_0^{ij} = \begin{pmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{pmatrix}, \tag{4.3}$$

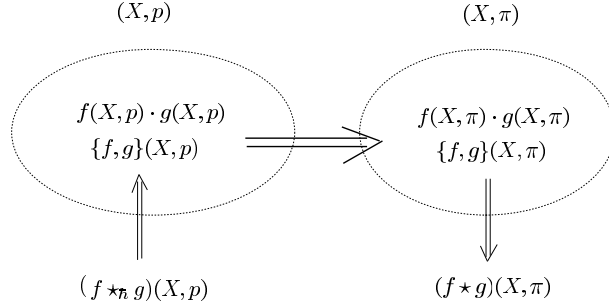


Fig. 2. This figure represents the derivation of the LHS of the gauge-covariant Dyson equation, where f and g are certain functions in the Dyson equation. The derivation has three steps. First, we obtain the Poisson structure of the Dyson equation with Wigner space (X, p) . This Poisson structure is obtained by using the gradient expansion,⁹⁾ which is represented by the symbol “ \Uparrow ”. Secondly, we perform the variable transformation, $p_\mu - qA_\mu(X) \mapsto \pi_\mu$, which is represented by the symbol “ \Rightarrow ”. Thirdly, explicitly gauge-covariant Dyson equations are given by using the deformational quantization method which is represented by the symbol “ \Downarrow ”.

and $\eta^{\mu\nu}$ is the Minkowski metric $\eta = \text{diag}\{-1, 1, 1, 1\}$.

Step. 2: Next, we introduce the transformation of variable (2.14) and (2.15). This step corresponds to the calculation of C_0 and C_1 in definition (3.2) in (X, π) space. Using the transformations (2.14) and (2.15), we can write the derivative of f and g as follows:

$$\begin{aligned}
& (\partial_{X^\mu} f(X, p))(\partial_{p_\mu} g(X, p)) - (\partial_{p_\mu} f(X, p))(\partial_{X^\mu} g(X, p)) \\
&= (\partial_{X^\mu} f(X, \pi))(\partial_{\pi_\mu} g(X, \pi)) - (\partial_{\pi_\mu} f(X, \pi))(\partial_{X^\mu} g(X, \pi)) \\
&\quad - q((\partial_{X^\mu} A_\nu(X))(\partial_{\pi_\nu} f(X, \pi))(\partial_{\pi_\mu} g(X, \pi)) - (\partial_{X^\nu} A_\mu(X))(\partial_{\pi_\mu} f(X, \pi))(\partial_{\pi_\nu} g(X, \pi))) \\
&= (\partial_{X^\mu} f(X, \pi))(\partial_{\pi_\mu} g(X, \pi)) - (\partial_{\pi_\mu} f(X, \pi))(\partial_{X^\mu} g(X, \pi)) + qF_{\mu\nu}(\partial_{\pi_\mu} f(X, \pi))(\partial_{\pi_\nu} g(X, \pi)),
\end{aligned} \tag{4.4}$$

where F is the field strength: $F_{\mu\nu} = \partial_{X^\mu} A_\nu - \partial_{X^\nu} A_\mu$. Accordingly the Poisson structure is transformed as:

$$\begin{aligned}
\alpha_0^{ij} &\mapsto \alpha^{ij} := \begin{pmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & qF^{\mu\nu}(X) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & qF^{\mu\nu} \end{pmatrix} \\
&\equiv \alpha_0^{ij} + \alpha_F^{ij}.
\end{aligned} \tag{4.5}$$

By this replacements, the zeroth-order terms and the first-order terms are transformed as

$$((\hat{\underline{G}}_0^{(0)-1}(\tilde{x}) - \hat{\underline{\Sigma}}(\tilde{x})) \cdot \hat{\underline{G}}(\tilde{x})), \tag{4.6}$$

$$\frac{i\hbar}{2} \left\{ (\hat{\underline{G}}_0^{(0)-1}(\tilde{x}) - \hat{\underline{\Sigma}}(\tilde{x})), \hat{\underline{G}}(\tilde{x}) \right\}, \tag{4.7}$$

where $\tilde{x} := (X, \pi)$. In (4.7), $\{f, g\} := \sum_{i,j} \alpha^{ij}(\partial_{\tilde{x}^i} f)(\partial_{\tilde{x}^j} g)$ is also the Poisson bracket, i.e., satisfies Jacobi's identity. Hereafter, the tilde in \tilde{x} will be omitted for simplicity.

4.2. Gauge-covariant formulation

Step 3: From the 0th-order and the first-order terms obtained in the previous section, gauge-covariant-Dyson equations are given by using the deformational quantization method. By constructing the star product from (4.6) and (4.7), the Dyson equation reads:

$$((\hat{\underline{G}}_0^{(0)-1}(X, \pi) - \hat{\underline{\Sigma}}(X, \pi)) \star \hat{\underline{G}}(X, \pi) = 1, \quad (4.8)$$

where the symbol “ \star ” is star product which is given the Poisson structure α with the Wigner space (X, π) .

Similarly,

$$\hat{\underline{G}}(X, \pi) \star ((\hat{\underline{G}}_0^{(0)-1}(X, \pi) - \hat{\underline{\Sigma}}(X, \pi)) = 1. \quad (4.9)$$

Note that the difference between Eqs. (2.7) and (4.8) is only in the definition of the product. Namely, to obtain the explicitly gauge-covariant Dyson equation, we only change the product “ \star_{\hbar} ” to “ \star ”.

Response to electromagnetic field

Let's recall the diagram rules for Kontsevich's star product. The rules relates a bidifferential operator to a graph $\Gamma \in G_n$. G_n is a set of n th-order graphs with respect to \hbar , and each graph $\Gamma \in G_n$ is constructed by $n+2$ vertices which are labeled as “1”, “2”, \dots , “ n ”, “ L ” and “ R ”, and by $2n$ edges which start from the vertex “ k ”, $k \neq L$ or R . The vertices which are labeled as “1”, “2”, \dots , “ n ”, correspond to α . The vertices which are labeled as “ L ” and “ R ” correspond to $\hat{\underline{G}}$ or $(\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}})$. We separate the Poisson structure into two parts $\alpha_0 + \alpha_F$ as (4.5) and expand $\hat{\underline{G}}$ and $(\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}})$ in terms of F :

$$\begin{aligned} \hat{\underline{G}} &= \hat{\underline{G}}_0 + \hat{\underline{G}}_F + \hat{\underline{G}}_{F^2} + \dots \\ (\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}}) &= (\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}}_0) - \hat{\underline{\Sigma}}_F - \hat{\underline{\Sigma}}_{F^2} - \dots, \end{aligned} \quad (4.10)$$

where subscript $_0$ and $_{F^k}$ represent zeroth order and k -th order terms with respect to F , respectively. Thus the contributions corresponding to each graph are further classified according to the order in F . Now, the vertices “1”, “2”, \dots , “ n ” correspond to α_0 or α_F . The numbers of α_F and α_0 are denoted by n_{α_F} and n_{α_0} , the vertices “ L ” and “ R ” correspond to f_{F^l} and g_{F^m} , receptivity, where f and g represent $\hat{\underline{G}}$ or $(\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}})$, and l, m are some nonnegative integers. The order of the graph with respect to F is given by $l + m + n_{\alpha_F}$. The Kontsevich's diagram rules applied to the present case read as

1. $n = n_{\alpha_F} + n_{\alpha_0}$ vertices “1”, “2”, \dots , “ n ” are put in the upper-half complex plan $H_+ := \{z \in \mathbf{C} | \text{Im}(z) > 0\}$ and two vertices “ L ” and “ R ” are put at 0 and 1, respectively.

2. Vertices “1”, “2”, \dots , “ n ” have two edges $(k, v_k^{1,2})$, $k = 1 \sim n$, $v_k^{1,2} = 1 \sim n$, L, R and $k \neq v_k^1, k \neq v_k^2$, where each edge (k, v) starts at vertex “ k ” and ends at any other vertices “ v ”.
3. The set of vertices and edges are written by Γ . G_n is defined by the set of all graph of $n + 2$ vertices and $2n$ edges. $G_n \in \Gamma$.
4. Vertices “1”, “2”, \dots , “ n_{α_F} ” correspond to α_F , vertices “ $n_{\alpha_F} + 1$ ”, “ $n_{\alpha_F} + 2$ ”, \dots , “ $n_{\alpha_F} + n_{\alpha_0} = n$ ” correspond to α_0 , and “ L ” and “ R ” correspond to f_{F^l} and g_{F^m} , respectively.
5. The operator $B_{\Gamma, \alpha}$ is constructed as follows. Edge (k, v) corresponds to the differentiation with respect to x acting on a function corresponding to vertex “ v ” ($l = 1 \sim n, l \neq k$), “ L ” or “ R ”. Thus a set of the vertex “ k ” and the two edges $(k, v_k^{1,2})$ corresponds to bidifferential operator $\sum_{ij} \alpha_{0, \text{or}, F}^{ij} \partial_{x^i}(h_{v_k^1}) \partial_{x^j}(h_{v_k^2})$, where h_v is the function corresponding to the vertex v .
6. The weight w_Γ of Γ is defined by

$$w_\Gamma := \frac{1}{n!(2\pi)^n} \int_{\mathcal{H}_n} \bigwedge_{k=1}^n \left(d\phi_{(k, v_k^1)}^h \wedge d\phi_{(k, v_k^2)}^h \right),$$

$$\phi_{(k, v)}^h = \frac{1}{2i} \text{Log} \left(\frac{(q - p)(\bar{q} - p)}{(q - \bar{p})(\bar{q} - \bar{p})} \right),$$

where p and q are coordinates of vertices “ k ” and “ v ” in H_+ , respectively, and \mathcal{H}_n is the space of configurations on n numbered pair-wise distinct points on H_+ : $\mathcal{H}_n = \{(p_1, \dots, p_n) | p_k \in H_+, p_k \neq p_l \text{ for } k \neq l\}$.

7. The r th-order star product in terms of F is given by

$$(f_{F^l} \star g_{F^m})_{F^r} = \sum_{n_{\alpha_0}=0}^{\infty} (i\hbar/2)^{n_{\alpha_F} + n_{\alpha_0}} \sum_{\Gamma \in G_{n_{\alpha_F} + n_{\alpha_0}}} w_\Gamma B_{\Gamma, \alpha}(f_{F^l}, g_{F^m}),$$

where $r = l + m + n_{\alpha_F}$.

Additional diagram rules:

- A1. Two edges of each vertex corresponding to α_F connect with both vertices “ L ” and “ R ”.
- A2. Edges of the vertices corresponding to α_0 do not connect with the vertices corresponding to α_0 , because η is the constant metric.
- A3. At least one edge of the vertex corresponding to α_0 connects with “ L ” or “ R ”.

To calculate $(f \star g)_{F^r} \big|_{\substack{f=f_{F^l} \\ g=g_{F^m}}}$, first we separate the graph Γ into Γ_{α_F} and Γ_{α_0} . Γ_{α_F} is the graph consisted by vertices corresponding to α_F , and “ L ” and “ R ”, and edges starting from these vertices. Γ_{α_0} is the rest of the graph Γ without Γ_{α_F} . Namely, Γ_{α_0} is the set of the vertices corresponding to α_0 which are labeled by “ k ” ($k = n_{\alpha_F} \sim (n_{\alpha_F} + n_{\alpha_0})$) and edges $(k, v_k^{1,2})$. Next, we calculate the weight $w_{n_{\alpha_F}}$ and the operator $B_{\Gamma_{\alpha_F}, \alpha_F}$ corresponding to Γ_{α_F} , and later those for Γ_{α_0} .

Separation of graph Γ

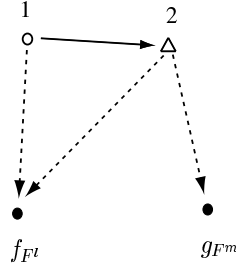


Fig. 3. A four vertices graph, where the white circle and the white triangle represent α_0 and α_F , respectively, and the dotted arrow and the real arrow represent the derivative with respect to π and X , respectively.

We now prove $w_\Gamma B_{\Gamma,\alpha} = w_{n_{\alpha_0}, n_{\alpha_F}} B_{\Gamma_{\alpha_0}, \alpha_0} \cdot w_{n_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}$, where $w_{n_{\alpha_0}, n_{\alpha_F}} = \frac{n_{\alpha_F}!}{(n_{\alpha_F} + n_{\alpha_0})!} \cdot \frac{w_1^{n_{\alpha_0}}}{n_{\alpha_0}!}$, $w_{n_F} = \frac{w_1^{n_{\alpha_F}}}{n_{\alpha_F}!}$ and w_1 is given by Eq. (3.11).

First we consider the case where the edges of the vertices corresponding to α_0 do not connect with α_F . In this case, $w_{n_{\alpha_0}, n_{\alpha_F}} = \frac{n_{\alpha_0}! n_{\alpha_F}!}{(n_{\alpha_F} + n_{\alpha_0})!} \cdot \frac{w_1^{n_{\alpha_0}}}{n_{\alpha_0}!}$. Secondly we consider the graph which consists of four vertices corresponding to α_0 , α_F , and “ L ” and “ R ” as shown in Fig. 3. We also assume that one edge of the vertex corresponding to α_0 connect with vertex corresponding to α_F . In this case, from additional diagram rule A3, another edge of the vertex has to connect with “ L ” or “ R ”. Since we can exchange the role “ R ” and “ L ” by the variable transformation $p \mapsto 1 - p$, ($p \in H_+$), we can assume that one edge of the vertex corresponding to α_0 connect with “ L ”. The weight w_Γ in this case is given by Eq. (3.12), i.e., the integrals for the weight is given by replacing coordinate of the vertex corresponding to α_F by coordinate of “ R ” (or “ L ”) in H_+ . From additional diagram rules, above result of integrals is applied to every graph. Namely, the position of each vertex corresponding to α_0 and α_F can be move independently in integrals, and the entangled integral does not appear. Therefore the weight $w_{n_{\alpha_0}, n_{\alpha_F}}$ of a graph Γ_{α_0} only depend on the number of vertices corresponding to α_0 and α_F . and $w_\Gamma = w_{n_{\alpha_0}, n_{\alpha_F}} \cdot w_{n_{\alpha_F}}$ holds generally. Finally, we can obtain $w_{n_{\alpha_0}} = \frac{n_{\alpha_F}! n_{\alpha_0}!}{(n_{\alpha_F} + n_{\alpha_0})!} \cdot \frac{w_1^{n_{\alpha_0}}}{n_{\alpha_0}!}$ and separate the operator $w_\Gamma B_{\Gamma,\alpha}$ into $w_{n_{\alpha_0}} B_{\Gamma_{\alpha_0}, \alpha_0}$ and $w_{n_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}$.

Explicit expression for star product

We will proceed to obtain the main result of this paper (Eq. (4.16)) for the gauge-covariant star product. We treat α_F , f_{F^l} and g_{F^m} as a cluster, and α_0 as an operator acting on the cluster as shown in Fig. 4. In this figure, the operator corresponding to the graph (a) is calculated by two steps. First, the graph (b) corresponding to $w_{n_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}$ is treated as a cluster, and is given by Kontsevich’s rules. Secondly, we calculate the operator $w_{n_{\alpha_0}} B_{\Gamma_{\alpha_0}, \alpha_0}$ that acts on the cluster as in graph (c), where the big open circle represents the graph (b). Namely,

$$(f \star g)_{F^r} \Big|_{\substack{f=f_{F^l} \\ g=g_{F^m}}} = \sum_{n_{\alpha_0}, \Gamma_{\alpha_0} \in G'} \left(\frac{i\hbar}{2} \right)^n w_{\Gamma_{\alpha_0}} B_{\Gamma_{\alpha_0}, \alpha_0} \cdot w_{\Gamma_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F} (f_{F^l}, g_{F^m}),$$

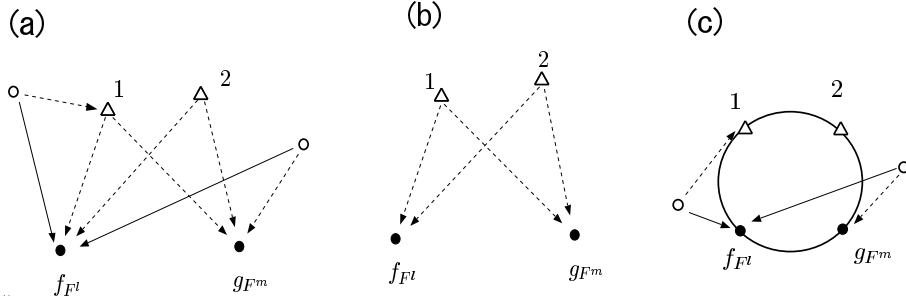


Fig. 4. This figure shows the calculation method of the graph (a), where the dotted arrow and real arrow represent the derivative with respect to π and X , respectively, and the white circle and the white triangle represent α_0 and α_F , respectively. We rewrite the graph (a) as the graph (c) which is given by the cluster represented by the big circle and the operators into it, where the big circle represents the graph (b).

(4.11)

where G' is the set of Γ_{α_0} .

We consider the contribution of the cluster. Since edges of $\alpha_F(X)$ correspond to the derivative with respect to π^μ , α_F -vertices do not connect with each other. In this case, the operator $w_{n_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}$ is given by

$$w_{n_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}(f_{F^l}, g_{F^m}) = \left(\frac{2}{i\hbar}\right)^{n_{\alpha_F}} \frac{1}{n_{\alpha_F}!} f_{F^l} \left(\frac{i\hbar}{2} \overleftarrow{\partial}_{\pi^\mu} q F^{\mu\nu} \overrightarrow{\partial}_{\pi^\nu}\right)^{n_{\alpha_F}} g_{F^m}. \quad (4.12)$$

Using Eq. (4.12), $(f \star g)_{F^r} \big|_{\substack{f=f_{F^l} \\ g=g_{F^m}}}$ is given by

$$\begin{aligned} (f \star g)_{F^r} \big|_{\substack{f=f_{F^l} \\ g=g_{F^m}}} &= \sum_{n, \Gamma \in G_n} \left(\frac{i\hbar}{2}\right)^n w_{n_{\alpha_0}, n_{\alpha_F}} B_{\Gamma_{\alpha_0}, \alpha_0} \cdot w_{n_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}(f_{F^l}, g_{F^m}) \\ &= \sum_{n_{\alpha_0}} \frac{(n_{\alpha_0} + n_{\alpha_F})!}{n_{\alpha_0}! n_{\alpha_F}!} \left(\frac{i\hbar}{2}\right)^{n_{\alpha_0} + n_{\alpha_F}} \frac{n_{\alpha_0}! n_{\alpha_F}!}{(n_{\alpha_0} + n_{\alpha_F})!} \frac{\hat{\alpha}_0^{n_{\alpha_0}}}{n_{\alpha_0}!} w_{\Gamma_{\alpha_F}} B_{\Gamma_{\alpha_F}, \alpha_F}(f_{F^l}, g_{F^m}) \\ &= \exp\left(\frac{i\hbar}{2} \hat{\alpha}_0\right) \left(\frac{1}{n_{\alpha_F}!} f_{F^l} \left(\frac{i\hbar}{2} \overleftarrow{\partial}_{\pi^\mu} q F^{\mu\nu} \overrightarrow{\partial}_{\pi^\nu}\right)^{n_{\alpha_F}} g_{F^m}\right), \end{aligned} \quad (4.13)$$

where $\hat{\alpha}_0$ is defined by

$$\hat{\alpha}_0(f_1, f_2, \dots) := \sum_{1 \leq k_1 < k_2 < \infty} \sum_{i, j} \alpha_0^{ij} f_1 \cdots (\partial_{x^i} f_{k_1}) \cdots (\partial_{x^j} f_{k_2}) \cdots. \quad (4.14)$$

Thus the r th-order terms $(f \star g)_{F^r}$ is given by

$$(f \star g)_{F^r} = \sum_{\substack{l, m \in \mathbf{N} \\ 0 \leq l+m \leq r}} (f \star g)_{F^r} \big|_{\substack{f=f_{F^l} \\ g=g_{F^m}}}. \quad (4.15)$$

Substituting Eqs. (4.13) and (4.15) into Eqs. (4.8) and (4.9), we obtain the gauge-covariant Dyson equations in the presence of $F^{\mu\nu}$ as

$$(f \star g)(X, \pi) = e^{\frac{i\hbar}{2}\hat{\alpha}_0}(f(X, \pi)e^{\frac{i\hbar}{2}\overleftarrow{\partial}_{\pi\mu}qF^{\mu\nu}\overrightarrow{\partial}_{\pi\nu}}g(X, \pi)). \quad (4.16)$$

When f , g and F are expanded by Fourier series, the operator $e^{\frac{i\hbar}{2}\hat{\alpha}_0}$ equals to the translation operator in the momentum space. We use this fact in the following section.

§5. Linear response theory

As an application of the formalism developed above, we discuss in this section the linear response to the electromagnetic field $F^{\mu\nu} \propto e^{iQ_\mu X^\mu}$ starting from the Dyson equation. The linear components of Eq. (4.8) read

$$((\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}}) \star \hat{\underline{G}})_F(X, \pi) = 0, \quad (5.1)$$

$$(\hat{\underline{G}} \star (\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}}))_F(X, \pi) = 0, \quad (5.2)$$

where $\hat{\underline{G}}_0^{(0)-1}(\pi) = \pi^0 - \hat{H}(\pi)$, \hat{H} is the Hamiltonian of electrons and the linear order terms in $F_{\mu\nu}$ can be written as

$$(f \star g)_F := (f_F \star_0 g_0) + (f_0 \star_0 g_F) + (f_0 \star_F g_0), \quad (5.3)$$

Each term is defined by

$$(f_F \star_0 g_0) := e^{\frac{i\hbar}{2}\hat{\alpha}_0}(f_F(X, \pi)g_0(X, \pi)), \quad (5.4)$$

$$(f_0 \star_0 g_F) := e^{\frac{i\hbar}{2}\hat{\alpha}_0}(f_0(X, \pi)g_F(X, \pi)), \quad (5.5)$$

$$(f_0 \star_F g_0) := e^{\frac{i\hbar}{2}\hat{\alpha}_0}(f_0(X, \pi)(\frac{i\hbar}{2}\overleftarrow{\partial}_{\pi\mu}qF^{\mu\nu}\overrightarrow{\partial}_{\pi\nu})g_0(X, \pi)), \quad (5.6)$$

where f and g represent $\hat{\underline{G}}$ or $(\hat{\underline{G}}_0^{(0)-1} - \hat{\underline{\Sigma}})$. Hereafter, we assume that the equilibrium system is uniform, and the nonequilibrium one is steady, i.e., $f_F(X, \pi), g_F(X, \pi) \propto e^{iQ_\mu X^\mu}$, $f_0(X, \pi) = f_0(\pi)$ and $g_0(X, \pi) = g_0(\pi)$. In this case, Eqs. (5.4), (5.5) and (5.6) turn into

$$(f_F \star_0 g_0) = f_F(X, \pi)g_0(\pi - Q/2), \quad (5.7)$$

$$(f_0 \star_0 g_F) = f_0(\pi + Q/2)g_F(X, \pi), \quad (5.8)$$

$$(f_0 \star_F g_0) = \frac{i\hbar}{2}f_0(\pi + Q/2)\overleftarrow{\partial}_{\pi\mu}qF^{\mu\nu}\overrightarrow{\partial}_{\pi\nu}g_0(\pi - Q/2), \quad (5.9)$$

where $e^{a_\mu\partial_{\pi\mu}}$ in $e^{\frac{i\hbar}{2}\hat{\alpha}_0}$ shifts the momentum π_μ to $\pi_\mu + a_\mu$. From Eq. (5.1), the equation of the Green functions is given by

$$\begin{aligned} \hat{\underline{G}}_F(X, \pi) &= \hat{\underline{G}}_0(\pi + Q/2)\hat{\underline{\Sigma}}_F(X, \pi)\hat{\underline{G}}_0(\pi - Q/2) \\ &\quad - \hat{\underline{G}}_0(\pi + Q/2)((\hat{\underline{G}}_0(\pi + Q/2)\overleftarrow{\partial}_{\pi\mu}qF^{\mu\nu}\overrightarrow{\partial}_{\pi\nu}\hat{\underline{G}}_0(\pi - Q/2)). \end{aligned} \quad (5.10)$$

We note that Eq. (5.10) is rewritten by

$$\begin{aligned}
(\hat{G}_F^< - (n_{\text{Fermi}} \star_0 \hat{G}_F^A - \hat{G}_F^R \star_0 n_{\text{Fermi}})) &= \hat{G}_0^R \star_0 (\hat{\Sigma}_F^< - (n_{\text{Fermi}} \star_0 \hat{\Sigma}_F^A - \hat{\Sigma}_F^R \star_0 n_{\text{Fermi}})) \star_0 \hat{G}_0^A \\
&+ \hat{G}_0^R \star_0 n_{\text{Fermi}} \star_0 ((\hat{G}_0^A)^{-1} \star_F \hat{G}_0^A) - \hat{G}_0^R \star_0 ((\hat{G}_0^R)^{-1} \star_F \hat{G}_0^R) \star_0 n_{\text{Fermi}} \\
&- \hat{G}_0^R \star_0 ((\hat{G}_0^R)^{-1} \star_F \hat{G}_0^<) + \hat{G}_0^R \star_0 \hat{\Sigma}_0^< \star_F \hat{G}_0^A,
\end{aligned} \tag{5.11}$$

where n_{Fermi} is the Fermi distribution function and $G_0^< = (G_0^A - G_0^R)n_{\text{Fermi}}$. We have used the retarded and the advanced components of Eqs. (5.1) and (5.2):

$$(\hat{G}_0^{R,A})^{-1} \star_0 \hat{G}_F^{R,A} + (\hat{G}_F^{R,A})^{-1} \star_0 \hat{G}_0^{R,A} + (\hat{G}_0^{R,A})^{-1} \star_F \hat{G}_0^{R,A} = 0, \tag{5.12}$$

$$\hat{G}_F^{R,A} \star_0 (\hat{G}_0^{R,A})^{-1} + \hat{G}_0^{R,A} \star_0 (\hat{G}_F^{R,A})^{-1} + \hat{G}_0^{R,A} \star_F (\hat{G}_0^{R,A})^{-1} = 0. \tag{5.13}$$

Therefore we obtain the following equations:

$$\hat{G}_F^I := \hat{G}_F^< - (n_{\text{Fermi}} \star_0 \hat{G}_F^A - \hat{G}_F^R \star_0 n_{\text{Fermi}}), \tag{5.14}$$

$$\hat{G}_F^{II} := n_{\text{Fermi}} \star_0 \hat{G}_F^A - \hat{G}_F^R \star_0 n_{\text{Fermi}}, \tag{5.15}$$

$$\hat{\Sigma}_F^I := \hat{\Sigma}_F^< - (n_{\text{Fermi}} \star_0 \hat{\Sigma}_F^A - \hat{\Sigma}_F^R \star_0 n_{\text{Fermi}}), \tag{5.16}$$

$$\hat{\Sigma}_F^{II} := n_{\text{Fermi}} \star_0 \hat{\Sigma}_F^A - \hat{\Sigma}_F^R \star_0 n_{\text{Fermi}}, \tag{5.17}$$

$$\begin{aligned}
\hat{G}_F^I &= \hat{G}_0^R \star_0 \hat{\Sigma}_F^I \star_0 \hat{G}_0^A \\
&+ \hat{G}_0^R \star_0 n_{\text{Fermi}} \star_0 ((\hat{G}_0^A)^{-1} \star_F \hat{G}_0^A) - \hat{G}_0^R \star_0 ((\hat{G}_0^R)^{-1} \star_F \hat{G}_0^R) \star_0 n_{\text{Fermi}} \\
&- \hat{G}_0^R \star_0 ((\hat{G}_0^R)^{-1} \star_F \hat{G}_0^<) + \hat{G}_0^R \star_0 \hat{\Sigma}_0^< \star_F \hat{G}_0^A.
\end{aligned} \tag{5.18}$$

Eq. (5.18) is the generalization of the Středa formula.^{13),15)}

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